

Space-time symmetric extension of non-relativistic quantum mechanics

Eduardo O. Dias* and Fernando Parisio†

Departamento de Física, Universidade Federal de Pernambuco, Recife, Pernambuco 50670-901, Brazil

In quantum theory we refer to the probability of finding a particle between positions x and $x + dx$ at the instant t , although we have no capacity of predicting exactly when the detection occurs. In this work, first we present an extended non-relativistic quantum formalism where space and time play equivalent roles. It leads to the probability of finding a particle between x and $x + dx$ during $[t, t + dt]$. Then, we find a Schrödinger-like equation for a “mirror” wave function $\phi(t, x)$ associated with the probability of measuring the system between t and $t + dt$, given that detection occurs at x . In this framework, it is shown that energy measurements of a stationary state display a non-zero dispersion, and that energy-time uncertainty arises from first principles. We show that a central result on arrival time, obtained through approaches that resort to *ad hoc* assumptions, is a natural, built-in part of the formalism presented here.

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In Schrödinger quantum mechanics (QM) there is a clear asymmetry between time and space. Time is a continuous parameter that can be chosen with arbitrary precision and used to label the solution of the wave equation. In contrast, the position of a particle is seen as an operator, and therefore its value under a measurement is inherently probabilistic. It is common to hear that this asymmetry is due to the non-relativistic character of the Schrödinger equation (SE). Although partially correct, this argument is largely insufficient to justify all the disparity between space and time in the formalism of QM.

A clear illustration is as follows. In a position measurement, $\psi(x, t) = \langle x | \psi(t) \rangle$ gives the probability amplitude of finding the particle within $[x, x + dx]$, *given that* the time of detection is t . Would it not be equally reasonable, even in the non-relativistic domain, to ask about the probability of measuring the particle between x and $x + dx$, and t and $t + dt$? In this broader scenario, inquiring about the state of a particle at a given time t (as we often do), should make as much sense as asking about the state of that particle in a given position x (which we never do). In addition, if symmetry is to hold at this level, then there should exist a “mirror” wave function $\phi(t, x) = \langle t | \phi(x) \rangle$, where x is a continuous parameter and t is the eigenvalue of an observable. If the location of particle becomes a physical reality only when a measurement is made, then it is a tenable position to expect that time should emerge in the same way. To earnestly consider these issues is the main goal of this manuscript.

Time has been addressed in different contexts in QM [1–23]. Common to several of these works is the attempt to remain within the borders of the standard theory. However, the solution to the arrival-time problem is considered by several authors to lay outside the framework of QM. It concerns the arrival of a particle in a spatially localized apparatus, where a time operator may be defined so that the relation $[\hat{T}, \hat{H}] = i\hbar$ is satisfied, and

the objective is to obtain the probability distribution for the detection times. This idea gave rise to numerous studies, e. g., in quantum tunnelling [6, 7] and lifetime of metastable systems. We will show that the standard arrival-time distribution, obtained by various approaches that usually resort to *ad hoc* assumptions, is a natural, built-in part of the formalism presented here.

Recent works [12, 17] building on a proposal by Page and Wootters [24], present specific similarities to this letter. As in reference [12], we also consider a time variable t with a Hilbert space \mathcal{H}_T isomorphic to that of a spinless particle in one dimension, \mathcal{H}_X . However, our approach does not assume that it is related to an external clock, or that it is a formal extension of a physical system. In our formalism \mathcal{H}_T (intrinsic to the system) is on the same footing as \mathcal{H}_X . Moreover, differently from Ref. [12], what we propose to be extended is the set of possible statistical inferences that QM is able to deal with.

Some symmetry between time and position in QM can be found, although often concealed by the standard presentation of the theory. An example is the pair of equations: $\hat{H}\hat{U}_t(t, t') = i\hbar(d/dt)\hat{U}_t(t, t')$, and $\hat{p}\hat{U}_x(x, x') = i\hbar(d/dx)\hat{U}_x(x, x')$, where \hat{H} is the Hamiltonian of the system, \hat{U}_t is the time evolution operator, \hat{p} is the momentum operator, and \hat{U}_x is the translation operator. At a formal level, there is a complete interplay between the pairs (\hat{p}, x) and (\hat{H}, t) . In spite of this perfect correspondence at a mathematical level, there is a physical asymmetry. In the first place, there is a clear lack of kets $|t\rangle$ satisfying $\hat{H}|t\rangle \stackrel{?}{=} (i\hbar d/dt)|t\rangle$, by analogy with $\hat{p}|x\rangle = (i\hbar d/dx)|x\rangle$. The existence of such equations, where x and t play formally similar roles, would only make sense if \hat{H} and \hat{p} were also considered on the same footing. This does not happen in standard QM because, while \hat{p} is defined by its action upon $\psi(x)$, \hat{H} comes from the replacement of x and p by \hat{X} and \hat{p} in the classical, symmetrized Hamiltonian $H(x, p)$.

Furthermore, in a formalism intending to promote time to a physical, observable quantity, the probability of finding a particle in $[x, x + dx]$ at an infinitely precise time

* corresponding author: eduardodias@df.ufpe.br

† parisio@df.ufpe.br

t must be rigorously zero. However, the probability density $\mathcal{P}(x, t)$ of finding the particle in the space and time intervals $[x, x + dx]$ and $[t, t + dt]$ is well-defined. These quantities should be related by

$$\mathcal{P}(x, t) dx dt = f(t) |\psi(x, t)|^2 dx dt, \quad (1)$$

where $f(t)$ is determined by Bayes rule, according to which $\mathcal{P}(x, t)$ is equal to the probability of finding the particle between x and $x + dx$ given that the measurement occurred precisely at t , $|\psi(x, t)|^2$, times the probability of the system being measured between t and $t + dt$, $f(t)$, whatever the outcome. For this reason, we express the wave function with the more appropriate notation $\psi(x|t)$.

It is essential to realize that the function $f(t)$ cannot be obtained through the knowledge of $|\psi(t)\rangle$, the solution of the SE. The temporal weighting function constitutes new information necessary to express the full state of the system. By “full state” we mean the information necessary to predict experimental outcomes related to statistical inferences other than those with a fixed time, that is, outside the traditional scope of QM.

In addition, due to the symmetry of Bayes rule, we can express the probability $\mathcal{P}(x, t) dx dt$ as $|\phi(t|x)|^2 g(x) dx dt$, where the quantities have analogous roles as those of Eq. (1), with $t \rightleftharpoons x$. Therefore, $|\phi(t|x)|^2 dt$ corresponds to the probability of finding the particle in the time window $[t, t + dt]$ given that the position measurement gives exactly x , and $g(x)$ is the distribution associated with position measurements regardless of t . Note that these quantities are not present in standard QM. Operationally, in order to determine $\mathcal{P}(x, t)$ one has to fill the space with detectors and turn them on simultaneously with respect to the laboratory clock, and wait until one of the detectors, located at some position x , clicks at some time t . This corresponds to an event registered at (x, t) . Then, repeat the procedure several times to extract the statistics described by $\mathcal{P}(x, t)$.

Knowing that quantum mechanics is a remarkably successful theory, and considering all the previous observations, we propose two *supplementary* assumptions to equip QM to deal with a broader set of statistical inferences (corresponding to valid experimental questions).

Assumption 1: The minimal Hilbert space necessary for a complete quantum description of a spinless particle in one dimension is $\mathcal{H} = \mathcal{H}_X \otimes \mathcal{H}_T$, where \mathcal{H}_T is as intrinsic to the system as \mathcal{H}_X . Accordingly, the ket that represents the full state of a quantum particle, denoted by $|\Psi\rangle \in \mathcal{H}$, can be expressed as

$$|\Psi\rangle = \int \int \Psi(x \& t) |x t\rangle dx dt, \quad (2)$$

with $|x t\rangle = |x\rangle \otimes |t\rangle$ and $\langle x' t' | \Psi \rangle = \Psi(x' \& t')$. We interpret $|x t\rangle$ as the state of a particle that is observed at the position x and at the instant t . We employed the notation $\Psi(x \& t)$ to make it clear that this quantity is *inequivalent* to the wave function $\psi(x, t) = \psi(x|t)$. The squared modulus of $\Psi(x \& t)$ is $\mathcal{P}(x, t)$ in Eq. (1). Bayes

rule for the amplitudes has the general form

$$\Psi(x \& t) = \psi(x|t) \sqrt{f(t)} e^{i\alpha(x, t)} = \phi(t|x) \sqrt{g(x)} e^{i\beta(x, t)}, \quad (3)$$

with $f(t) \geq 0$ and $g(x) \geq 0$ (we simply set $\alpha = \beta = 0$). For a complete description we need either the first or the second equality in Eq. (3), not both. We stress that in an experiment where the statistics of some observable is done by selecting a specific time, all the usual results of QM immediately follow by just knowing $\psi(x|t)$. On the other hand, if the position of the measurement is fixed as a conditional parameter, as it happens in arrival-time experiments, one has just to know $\phi(t|x)$ to predict their results. Finally, whenever the position x and the time t (both unconstrained) are under measurement, the results should be given by the wave function $\Psi(x \& t)$. It is worth mentioning that in previous approaches to the role of time in QM, expressions which are notationally similar to the space-time integral (2) have appeared. For example, Eq. (6) of [21] has completely different construction and interpretation, since it corresponds to a new representation of ordinary states of QM. This state is defined in order to take into account the time interval of interaction with the measuring device, and it can be derived, in principle, by using the SE, as is done in Refs. [21, 22]. On the other hand, Eq. (2) encompasses a temporal probabilistic character not obtainable from the traditional formulations.

By writing Eq. (3) as $|\psi(x|t)|^2 / |\phi(t|x)|^2 = g(x) / f(t)$ and integrating over x we obtain

$$f(t) = \left[\int |\psi(x|t)|^2 / |\phi(t|x)|^2 dx \right]^{-1}, \quad (4)$$

for $\phi(t|x) \neq 0$ and $f(t) \neq 0$. This result makes it clear that the temporal distribution does not depend on the details of the detectors.

Assumption 2: By analogy with the standard operators \hat{X} , \hat{p} , and $\hat{H}(\hat{X}, \hat{p}; t)$ acting in \mathcal{H}_X , we define the mirror operators: observation time \hat{T} (mirror of \hat{X}), Hamiltonian \hat{h} (mirror of \hat{p}), and momentum $\hat{P}(\hat{T}, \hat{h}; x)$ (mirror of \hat{H}) acting in \mathcal{H}_T . The observables in lower cases are solely defined by their action upon the bases $\{|x\rangle\}$ and $\{|t\rangle\}$, through the relations

$$\frac{\langle x t | \hat{p} | \Psi \rangle}{i\hbar} = -\frac{\partial}{\partial x} \Psi(x \& t), \quad \frac{\langle x t | \hat{h} | \Psi \rangle}{i\hbar} = \frac{\partial}{\partial t} \Psi(x \& t). \quad (5)$$

These operators are canonically conjugated to \hat{X} and \hat{T} , respectively, which are defined by $\hat{X}|x t\rangle = x|x t\rangle$ and $\hat{T}|x t\rangle = t|x t\rangle$. The inverse Fourier transform of $|x t\rangle$ reads $|x t\rangle = 1/(2\pi\hbar) \int \int \exp(-ipx/\hbar + i\varepsilon t/\hbar) |p \varepsilon\rangle dp d\varepsilon$ with $|p \varepsilon\rangle = |p\rangle \otimes |\varepsilon\rangle$, $|p\rangle \in \mathcal{H}_X$ and $|\varepsilon\rangle \in \mathcal{H}_T$. This implies $|\Psi\rangle = \int \int \tilde{\Psi}(p \& \varepsilon) |p \varepsilon\rangle dp d\varepsilon$, where $\tilde{\Psi}(p \& \varepsilon) = \frac{1}{2\pi\hbar} \int \int \exp[-i(px - \varepsilon t)/\hbar] \Psi(x \& t) dx dt$.

The expectation value of energy, e.g., is given by an average over the whole Hilbert space $\mathcal{H} = \mathcal{H}_X \otimes \mathcal{H}_T$:

$$\langle h \rangle = \int \int |\tilde{\Psi}(p \& \varepsilon)|^2 \varepsilon dp d\varepsilon, \quad (6)$$

an analogous relation holding for the linear momentum. Note that these averages are given by the lower case observables. Definition (6) leaves mean values of energy unchanged while variances may change with respect to the standard theory. This is not in contradiction with QM since Eq. (6) is defined in a different way from the mean value of traditional QM, where time is fixed. Here, we take into account an intrinsic probabilistic character of the detection moment and, consequently, an extra integration over time is necessary [see Eq. (2)]. A compelling consequence of these relations is that, whenever a system is under measurement and it is not possible/desirable to fix the time, there is a nonzero variance associated with its energy, even if its state is stationary, as we will verify in what follows.

Consider the general problem of a confined particle with a Schrödinger state being: $|\psi(t)\rangle = \exp(-iE_n t/\hbar) |\psi_n\rangle$, where $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$. To derive $f(t)$, suppose that the particle can be detected all over the region where the wave function is non-vanishing. If the particle has a finite probability q of being measured after $t = 0$ and before $t = \delta t$, but it turns out that the detection did not happen, for a Markov process, the probability of it occurring between $t = \delta t$ and $t = 2\delta t$ is the same as it was in $t = 0$. Thus, the probability of no observation up to $t = n\delta t$ is $P(t = n\delta t) = (1 - q)^n$. If δt is sufficiently small, we can assume $q \ll 1$, so that $P(t = n\delta t) \simeq e^{-n\delta t\Lambda} = e^{-\Lambda t}$, where we defined $\Lambda \equiv q/\delta t$. The associated probability density is $f(t) = -dP/dt$, leading to the Poisson distribution $f(t) = \Lambda e^{-\Lambda t}$. Combining Eq. (3) and the previous result, the complete state of the system can be written as

$$|\Psi_n\rangle = |\psi_n\rangle \otimes \int \sqrt{\Lambda} e^{(-\Lambda t/2 - iE_n t/\hbar)} |t\rangle dt. \quad (7)$$

The expectation time associated with the occurrence of the observation is easily obtained and reads $\langle T \rangle = 1/\Lambda$, as it should do. The time uncertainty is $\Delta T = \sqrt{\langle T^2 \rangle - \langle T \rangle^2} = 1/\Lambda$. Complementarily, we have

$$\tilde{\Psi}(p \& \varepsilon) = \sqrt{\Lambda} \int \tilde{\psi}_n(p) e^{i[(\varepsilon - E_n)/\hbar + i\Lambda/2]t} dt, \quad (8)$$

leading to the energy-momentum probability density given by $|\tilde{\Psi}(p \& \varepsilon)|^2 = |\tilde{\psi}_n(p)|^2 |\chi(\varepsilon)|^2$, with

$$|\chi(\varepsilon)|^2 = \frac{1}{\pi} \frac{\hbar\Lambda/2}{(\varepsilon - E_n)^2 + (\hbar\Lambda/2)^2}. \quad (9)$$

Note that $\tilde{\Psi}(p \& \varepsilon)$ is factorable because the initial wave function is a stationary state [25]. By replacing $|\tilde{\Psi}(p \& \varepsilon)|^2$ into Eq. (6), we have $\langle h \rangle = \int |\chi(\varepsilon)|^2 \varepsilon d\varepsilon$. Thus, the result of an energy measurement is ε , satisfying the Lorentzian distribution (9).

The distribution $|\chi(\varepsilon)|^2$ does not have a well-defined variance due to its fat tails. However, its full width at half maximum, $\delta\varepsilon$, is $\hbar\Lambda$. Thus, $\Delta T \delta\varepsilon \sim 1/\Lambda \times \hbar\Lambda = \hbar$. Result (9) predicts an energy linewidth similar to the natural linewidth which arises from the interaction with

the electromagnetic vacuum. In both cases the profile is Lorentzian, but the physical origins are completely different, since, in our formalism the linewidth appears because the detection time is considered as a probabilistic variable. Since linewidth measurements do not constrain the observation time, our formalism should apply and Eq. (8) would give the actual profile as a convolution of the Lorentzian in (9) with that describing the natural linewidth. The result is also a Lorentzian, but broader (width $\Lambda + \Gamma$) and with a lower peak [height $1/\pi(\Lambda + \Gamma)$], where Γ is the spontaneous decay rate. Therefore, if one is able to measure the natural linewidth minimizing all other broadening effects (Doppler effect, collisional effect, etc), then, the measured width should be *larger* than that predicted by QM (e. g., via *ab initio* calculations). Because Λ may be small, we may need to address situations for which Γ and Λ do not differ by more than a few orders of magnitude. Thus, it would be easier to observe this possibly subtle difference, if it exists, in long-lived systems (narrow linewidths).

We now derive the dynamic equation for $\phi(t|x)$, “dynamic” meaning how ϕ changes with x . We will do it through a direct analogy with SE. Let us define $|\phi(x)\rangle \in \mathcal{H}_T$, where x is a parameter that can be chosen arbitrarily in the same way as the time t in the standard theory. In addition, the ket $|t\rangle$ corresponds to the relative state of a particle that is observed at time t , so that $\langle t|\phi(x)\rangle = \phi(t|x)$.

Due to the isomorphism between \mathcal{H}_X and \mathcal{H}_T , we must have $\langle t|t'\rangle = \delta(t - t')$ and $\mathbb{I} = \int |t\rangle\langle t| dt$, the orthogonality of $\{|t\rangle\}$ ensuring that the particle is observed at a specific time. With these definitions, we can write $|\phi(x)\rangle = \int \phi(t|x) |t\rangle dt$ similarly to $|\psi(x)\rangle = \int \psi(x|t) |x\rangle dx$. To make sure that the particle will be observed during the measurement process, the state has to be normalized, $\langle\phi(x)|\phi(x)\rangle = 1$, which implies $\int |\phi(t|x)|^2 dt = 1$. Finally, we interpret $|\phi(t|x)|^2 dt$ as the probability of measuring the particle in the time interval $[t, t + dt]$, given that it is observed at the position x .

We proceed by attributing to the momentum operator \hat{P} acting in \mathcal{H}_T the same status and role as \hat{H} in conventional QM. Also, we use Eq. (5) to write $\langle t'|\hat{h}|t\rangle = \delta(t - t') (i\hbar d/dt)$, which automatically leads to the canonical commutation relation $[\hat{h}, \hat{T}] = i\hbar$. In addition, recall that SE describes how $|\psi(t)\rangle$ changes under “time translations”: $\hat{H}|\psi(t)\rangle = i\hbar(d/dt)|\psi(t)\rangle$, with $\hat{H} = \hat{p}^2/(2m) + \hat{V}(\hat{X}, t)$. The analogous relation for kets in \mathcal{H}_T is the space-dependent SE:

$$\hat{P}|\phi(x)\rangle = i\hbar \frac{d}{dx} |\phi(x)\rangle, \hat{P} = \pm \sqrt{2m [\hat{h} - \hat{V}(x, \hat{T})]}. \quad (10)$$

Since \hat{P} has two branches (signs \pm), we assume that $\phi(t|x)$ is a two-component pseudospinor:

$$\phi(t|x) = \begin{pmatrix} \phi^+(t|x) \\ \phi^-(t|x) \end{pmatrix}, \quad (11)$$

and the mirror equation is in fact

$$\hat{\sigma}_z \sqrt{2m \left[i\hbar \frac{d}{dt} - V(x, t) \right]} \phi(t|x) = i\hbar \frac{d}{dx} \phi(t|x), \quad (12)$$

where $\hat{\sigma}_z = \text{diag}(+1, -1)$. It is then clear that \sqrt{g} in Eq. (3) is to be understood as a vector with components \sqrt{g}^\pm . Equation (12), one of our central results, is a Schrödinger-like equation for $\phi(t|x)$, where the position x of the observation is a conditional parameter. Finally, we define in the usual way the associated probability density as $\rho = |\phi(t|x)|^2 \equiv \phi^\dagger(t|x)\phi(t|x)$.

Pauli pointed out the impossibility of defining a self-adjoint time operator conjugated to a Hamiltonian with a spectrum bounded from below [26]. Pauli's result is a no-go theorem constraining the possible time observables derived by using standard quantum theory. The suggested framework does not suffer from this limitation since it is clearly not contained in traditional QM. Moreover, it is not necessary for \hat{h} to be bounded from below since it plays the same role as \hat{p} in standard quantum theory. Positive values of energy may be required as a consequence of the boundary conditions on $\phi(t|x)$. In these circumstances, the commutation relation $[\hat{h}, \hat{T}] = i\hbar$ automatically leads to the energy-time uncertainty $\Delta\varepsilon\Delta T \geq \hbar/2$, where $\Delta\varepsilon$ and ΔT are the root-mean-square deviations of \hat{h} and \hat{T} , which act in \mathcal{H}_T . Note that, here, it is the customary relation $[\hat{P}, \hat{x}] = i\hbar$ that cannot be derived, since x is a parameter.

Hereafter, we focus on the wave equation for the free particle [$V(x, t) = 0$]. By inspecting Eq. (12), we can obtain the temporal eigenfunction for the momentum operator defined as $\hat{P}\phi_P(t) = P\phi_P(t)$, where $\hat{P} = \hat{\sigma}_z \sqrt{2m(i\hbar d/dt)}$. It is worth noting the analogy between the eigenfunction $\phi_P(t)$ (a space-independent function) and the time-independent Schrödinger state $\psi_E(x)$, which satisfies $\hat{H}\psi_E(x) = E\psi_E(x)$. In this latter case, the eigenenergy solution is simply $\psi_E(x|t) = \psi_E(x)\exp(-iEt/\hbar)$. Accordingly, we write $\phi(t|x) \equiv \phi_P(t|x) = \phi_P(t)\exp(iPx/\hbar)$. By substituting the previous definition into Eq. (12), we have $\hat{\sigma}_z \sqrt{2m(i\hbar d/dt)} \phi_P(t) = P\phi_P(t)$. The last step is to identify $\sqrt{d/dt}$ with the Riemann-Liouville fractional derivative ${}_{-\infty}D_t^{1/2}$, which is equivalent to the Caputo fractional derivative [27]. This leads to

$$\hat{\sigma}_z \sqrt{2mi\hbar} {}_{-\infty}D_t^{1/2} \phi_P(t) = P \phi_P(t). \quad (13)$$

Let us consider a solution such as $\phi_P^\pm(t) = C_P^\pm \exp(-iwt)$, and use the identity ${}_{-\infty}D_t^{1/2} \exp(-iwt) = \sqrt{-iw} \exp(-iwt)$. By doing this, we readily obtain the dispersion relation $P = \pm\sqrt{2m\hbar w}$, where we did not use negative energies since they lead to imaginary P .

Because Eq. (12) is linear, the general solution is $\phi^\pm(t|x) = \int_0^\infty A_P^\pm \exp(-iE_P t/\hbar) \exp(\pm iPx/\hbar) dP$, $E_P \equiv \hbar w = P^2/2m$. By setting $A_P^\pm \equiv C_P^\pm \sqrt{|P|/2\pi m}$, the temporal normalization condition for ρ reads

$\int_{-\infty}^\infty \rho(t|x) dt = 1 \Rightarrow \int_{-\infty}^\infty (|C_P^+|^2 + |C_P^-|^2) dP = 1$, where $|C_P^\pm|^2$ corresponds to the probability density of finding the particle with momentum $\pm P$. With this, we can express the general solution as

$$\phi(t|x) = \frac{1}{\sqrt{2\pi m\hbar}} \int_0^\infty \left(C_P^+ \sqrt{P} e^{iPx/\hbar} + C_P^- \sqrt{P} e^{-iPx/\hbar} \right) e^{-iE_P t/\hbar} dP.$$

It is natural to define the states $|P^\pm\rangle$ so that $\langle t|P^\pm\rangle = \phi_P^\pm(t) \equiv \sqrt{P/2\pi m\hbar} \exp(-iE_P t/\hbar)$. In this way, we can write $\phi^\pm(t|x) = \int C_P^\pm \phi_P^\pm(t) \exp(\pm iPx/\hbar) dP$. These states are the same eigenstates (with positive and negative momentum) as the time-of-arrival operator defined by adding the symmetrization and quantization of the classical expression $mx_{\text{class}}/p_{\text{class}}$ to conventional QM. This operator was first defined by Aharonov and Bohm [28], and later used by several other authors (see, for instance, [3, 10, 29]).

Note the symmetry between the general solutions for $\psi(x|t)$ and $\phi(t|x)$: $\phi^\pm(t|x) = \int C_P^\pm \phi_P^\pm(t) \exp(\pm iPx/\hbar) dP$ and $\psi(x|t) = \int C_E \psi_E(x) \exp(-iEt/\hbar) dE$. The function $\phi(t|x)$ is a superposition of momentum eigenstates, whereas $\psi(x|t)$ corresponds to a linear combination of states with well-defined energy. With this analogy, we can interpret $|\phi_P(t)|^2$ as the probability density of observing the particle at the instant t , given that it has momentum P , while $|\psi_E(x)|^2$ is the probability density of finding the particle at the position x , given that the energy is E . A fundamental issue related to time in QM is to seek a temporal distribution that describes the instant of time at which a certain property of a system assumes a given value [30]. This is exactly the interpretation of $|\phi_P(t)|^2$ with the momentum P as the physical property.

Finally, the probability density of finding the particle at the instant t , given that a measurement is performed at the position x , is given by

$$\rho(t|x) = \frac{1}{2\pi m\hbar} \left\{ \left| \int_0^\infty C_P^+ \sqrt{P} e^{iPx/\hbar - iE_P t/\hbar} dP \right|^2 + \left| \int_0^\infty C_P^- \sqrt{P} e^{-iPx/\hbar - iE_P t/\hbar} dP \right|^2 \right\}. \quad (14)$$

This is exactly the time-of-arrival probability density obtained by several authors via different approaches [8, 30–33] and is in excellent agreement with numerical “quantum jump” time-of-flight simulations [18]. However, the models employed have faced problems even in the free particle case. In Refs. [30, 32], e. g., Eq. (14) was obtained from the Schrödinger current density, which is not positive definite. It has been argued that there were *ad hoc* assumptions not included in standard quantum theory [34], e. g., the association of the signs in $\pm P$ with the direction of arrival [35]. Moreover, the time operator put forward by Aharonov and Bohm is semiclassical and system-dependent, since it is obtained by the quantization of the classical time of arrival at a certain point

x . Here, on the other hand, we develop a wave dynamics where temporal probability densities arise from first principles [the dynamic equation (12)] through a positive definite quantity ρ , similarly to $|\psi(x|t)|^2$ in standard QM. The distribution ρ is obtained with no dependence on the properties of a particular measuring apparatus.

We argue that part of the asymmetry between position and time in non-relativistic quantum mechanics is due to the fact that we experience these degrees of freedom in drastically different ways, and not only because of the lack of relativistic covariance. We tend to face time as a parameter much more naturally than position, although this inclination is not justifiable, on logical grounds. Guided by the requirement of symmetry between x and t as statistical variables and by Bayes theorem, we find a mirror wave function that gives new physical information, not obtainable through the knowledge of the Schrödinger wave function. We find the corresponding equation of motion, and show that the arrival-time distribution follows naturally. In contrast, previous derivations resort to assumptions which demand that one either gives up on the hermiticity of the operator \hat{T} or on the validity of its canonical commutation relation

with the Hamiltonian [30]. Here we kept both desirable properties, and found that actual measurements on an energy eigenstate lead to results with a non-zero dispersion, which illustrates how the energy-time uncertainty arises in the formalism.

Note the nature of the supplementation we propose. In situations where time is, in any way, fixed, standard QM emerges unchanged. However, QM does not provide any obvious answer to valid experimental questions related to other kinds of inference. We believe that the reason is made clear in the present work, which also provides a plausible fill to this gap.

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